Double-negation Shift as a constructive principle

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***Abstract*—We consider the Double-negation Shift (DNS) as a constructive principle in its own right and its effect on modified realizability (MR) and Dialectica (D) interpretations.**

We notice that DNS proves its own MR-interpretation, meaning that *a priori* one does not have to consider the more complex D-interpretation with Bar Recursion for interpreting Analysis. From the "with truth" variant of MR, we obtain the closure under Weak Church's Rule. We notice, in contrast, that DNS proves the double negation of the Limited Principle of Omniscience (LPO), hence of the solvability of the Halting Problem, and recall the related fact that DNS refutes a formal version of Church's Thesis (CT). This shows that intuitionistic Arithmetic plus DNS presents a distinct variant of Constructive Mathematics.

We revisit the standard approach of Kreisel and Spector for showing consistency of Analysis, and show that one can omit the preliminary double-negation translation phase.

Finally, we formalize the proofs of not-not-LPO and not-CT using a previously introduced constructive logic based on delimited control operators from the theory of programming languages.

I. Introduction

The logical principle known as Double-negation Shift,

$$\forall n \in \mathbb{N} \neg\neg A(n) \to \neg\neg \forall n \in \mathbb{N} A(n), \qquad \text{(DNS)}$$

that is not provable by intuitionistic logic, has first been isolated by Sigekatu Kuroda [1]. Its importance for the foundational program has been realized by Kreisel [2], who showed that if one can prove the Double-negation Shift, then one can prove the double-negation translation of the countable Axiom of Choice (AC^0). Coupled with Gödel's double-negation translation, that reduces Classical Arithmetic to Intuitionistic Arithmetic [3], and with Gödel's functional interpretation [4], [5], [6], that interprets Intuitionistic Arithmetic by functionals defined by primitive recursion at higher types (System T), a functional interpretation of DNS allows to extend this reduction (and relative consistency proof) to Classical *Analysis*. The functional interpretation of DNS has been given by Kreisel and Spector [7] by adding an additional recursion schema to System T, called Bar Recursion.

This method, functional interpretation with bar recursion, has been the main one when it comes to extracting computational content from proofs that use both classical logic and choice, and has had many applications, notably in the Proof Mining approach of Kohlenbach [8].

Recently, the author has shown, based on initial observations of Hugo Herbelin, that one can prove DNS in a system for first-order logic, based on the delimited control operators shift and reset of Danvy and Filinski [9], [10] from Programming Languages Theory. This system of pure logic, in the absence of all axioms, has been shown to be constructive.

The purpose of this paper is to investigate the meta-mathematical status of DNS in presence of higher-type Intuitionistic Arithmetic (HA^ω).

In Section II we revise the basic relationships between Double-negation Shift, the Countable Axiom of Choice, and double-negation translations, in the context of HA^ω.

In Section III, we show that DNS proves its own modified realizability interpretation, meaning that modified realizability is a viable (and simpler) alternative to functional interpretation and bar recursion, for extracting programs from proofs in Classical Analysis. This has practical implications for proof assistants based on intuitionistic logic that extract programs by modified realizability (like Coq).

Furthermore, we show from the "with truth" variant of modified realizability that HA^ω+AC^0+DNS has the constructive closure properties of HA^ω, among which a version of Church's Rule. However, DNS is known to refute (formal versions of) Church's Thesis. We briefly review this argument, that goes back to Gödel, and we argue that HA^ω+AC^0+DNS is a variety of Constructive Mathematics that permits to see in a new light the diagonalization arguments that reduce a problem to the non-solvability of the Halting Problem.

In Section IV, we also revisit the approach of functional interpretation and bar recursion. We observe, for the first time (to the best of our knowledge), that it is not necessary to perform a double-negation translation phase when interpreting Classical Analysis proofs, that is, that bar recursion already has some kind of double-negation translation built-in.

In Section V, we apply the system from [11] to formalize the arguments relating to Church's Thesis and the Halting Problem, by constructive delimited control operators directly.

In Section VI, we mention related works, that have not already been mentioned in the preceding sections, and we give some directions for future work.

We hope that this work will contribute to the program of enriching current proof assistant with constructive principles that are beyond intuitionistic logic, and with programming principles (control operators) that are beyond side-effect-free programming.

II. Preliminaries

In this paper, we will work in the context of higher-type intuitionistic, or Heyting, Arithmetic abbreviated HA^ω. Intuitively, one can think of HA^ω as a logical system based on

two orthogonal typed lambda calculi. At one level, we have the typed lambda calculus that is used to denote natural deduction proofs; this is the logical level, and we call the corresponding λ-terms: *proof terms*. At the other level, there is the typed lambda calculus that constructs individuals of the domain of quantification (the higher-types level), and the corresponding λ-terms are called just *terms*[1]. Hence, HA$^\omega$ stops short of a dependent type theory, which collapses the two levels into one. HA$^\omega$ is also closely connected to Gödel's System T [5], a quantifier-free logical theory based on equations between numerical terms constructed by higher-type primitive recursion.

Technically, HA$^\omega$ has as basis a multi-sorted first-order logic, with sorts built from 0 (standing for $\mathbb{N}$) and the arrow $\to$ (used to build functions and functionals over $\mathbb{N}$). These sorts are called *higher types* (or just *types*). Quantifiers $\forall, \exists$ are then tied to specific types; we usually suppress writing the type of a quantifier, but, when not clear from the context, the variable that is quantified over is annotated by a type in superscript. We use as type variables the symbols ρ, σ, τ.

For the *proof terms* we will use the concrete syntax from the upper box of Table I, ignoring for the moment all $\diamond$ annotating the $\vdash$.

The *terms* live in a lambda calculus with arrow, product and unit types. We have lambda abstraction $\dot{\lambda} x.t$ and application ts concerning the arrow type. Using pairs (t, s) we construct members of the pair type, and using the projections t_1 and t_2, we destruct them.[2] The constructor of the unit type is denoted tt.

There is a decidable equality relation $=_0$ between terms of type 0 (numbers). The Peano axioms are stated in terms of this equality, including the induction axiom schema for type 0 individuals. For the purposes of this paper, we do not need to be more precise than this, for more information the reader may take a look at a standard reference book like [12] or [8].

We will consider the following principles, both unprovable in HA$^\omega$. The Double-negation Shift at type ρ,

$$\forall x^\rho \neg\neg A(x) \to \neg\neg \forall x^\rho A(x), \qquad (\mathrm{DNS}^\rho)$$

and the Axiom of Choice at types ρ and σ,

$$\forall x^\rho \exists y^\sigma A(x,y) \to \exists f^{\rho\to\sigma} \forall x A(x, f(x)). \qquad (\mathrm{AC}^{\rho\sigma})$$

Particularly important will be the instances DNS^0 and AC^0 (short for AC$^{0\sigma}$). We will write DNS$^\omega$ and AC$^\omega$ when we intend to have DNS$^\rho$and AC$^{\rho\sigma}$ for all ρ and σ.

Although both DNS^0 and AC^0 are unprovable in HA$^\omega$, the later is admissible for arithmetical formulas, that is, if HA$^\omega$+AC^{0}proves an arithmetical A, then HA$^\omega$alone proves A [13]. A formula is *arithmetical* if it contains only type 0 quantifiers.

HA$^\omega$ extended with the Law of Excluded Middle is called PA$^\omega$, higher-type classical, or Peano, Arithmetic. The relation between HA$^\omega$ and PA$^\omega$ is given by double-negation translation.

[1]In standard presentations, the terms level is given by a combinator calculus, but a λ-calculus treatment is also possible, see for example 1.8.4. of [12].

[2]This unusual notation for writing projections is quite efficient when used in sections III and IV

Definition 1 (Kuroda's double-negation translation [1], or the call-by-value continuation-passing style translation [14])**.** The *double-negation translation of* A is the formula $A^\perp := \neg\neg A_\perp$, where $A_\perp$ is defined by recursion on the complexity of A by the clauses:

$$\begin{aligned}
P_\perp &:= P \qquad P\text{-prime}\\
(A \wedge B)_\perp &:= A_\perp \wedge B_\perp\\
(A \vee B)_\perp &:= A_\perp \vee B_\perp\\
(A \to B)_\perp &:= A_\perp \to B^\perp\\
(\forall x A(x))_\perp &:= \forall x (A^\perp(x))\\
(\exists x A(x))_\perp &:= \exists x (A_\perp(x))
\end{aligned}$$

Fact 1 ([12])**.** *In PA$^\omega$, $\vdash A \leftrightarrow A^\perp$. Also, $\Gamma \vdash A$ in PA$^\omega$ if and only if $\Gamma_\perp \vdash A^\perp$ in HA$^\omega$, where $\Gamma_\perp$ stands for the context obtained by applying the $(\cdot)_\perp$-translation to all formulas in Γ.*

Fact 2 (Lemma 1.10.9 of [12])**.** *In presence of DNS$^\omega$, HA$^\omega$ can prove the equivalence*

$$\neg\neg A \leftrightarrow A^\perp$$

between the double-negation of A and the double-negation translation of A. DNS^0 suffices to prove the same statement for arithmetical formulas A.

Fact 3. *If A is an instance of AC$^{\rho\sigma}$, then $\vdash A_\perp$ (and therefore also $\vdash A^\perp$) is derivable in HA$^\omega$+AC$^{\rho\sigma}$+DNS$^\rho$.*

Proof: The proof term

$$\lambda a.\lambda k.d(\tilde{\lambda} x.\lambda k'.ax(\lambda a'.\mathsf{dest}\ a'\ \mathsf{as}\ (x.e)\ \mathsf{in}\ k(x,e))) \\ (\lambda b.k(cb))$$

derives

$$\forall x^\rho \neg\neg \exists y^\sigma A_\perp(x,y) \to \neg\neg \exists f \forall x \neg\neg A_\perp(x, f(x)),$$

when the proof terms d and c derive the following instances of DNS$^\rho$ and AC$^{\rho\sigma}$:

$$\begin{aligned}
d &: \forall x \neg\neg \exists y \neg\neg A_\perp(x,y) \to \neg\neg \forall x \exists y \neg\neg A_\perp(x,y)\\
c &: \forall x \exists y \neg\neg A_\perp(x,y) \to \exists f \forall x \neg\neg A_\perp(x, f(x))
\end{aligned}$$

■

We call the theory PA$^\omega$+AC^0, *Classical Analysis*.

III. Modified realizability interpretation of DNS

Kreisel's modified realizability interpretation [15] can be seen as a precise statement of the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic, although its original aim was to show the non-derivability of Markov's Principle in Heyting Arithmetic. It is this realizability interpretation that is behind the program extraction facilities of proof assistants based on intuitionistic logic like Coq.

Let us recall the definition and basic properties of modified realizability relevant to this paper. The proofs of the facts may be found in standard references as [12], [16], [8]

Definition 2. For a term t and a formula A, the relation $t\ \mathsf{mr}\ A$ (t *interprets* A *by modified realizability*) is defined by recursion

on the complexity of A, by the following clauses:

$$\begin{aligned}
t \mathsf{\ mr\ } P &:= P \qquad \text{if } P\text{-prime}, t = \mathrm{tt}\\
t \mathsf{\ mr\ } A \wedge B &:= (t_1 \mathsf{\ mr\ } A) \wedge (t_2 \mathsf{\ mr\ } B)\\
t \mathsf{\ mr\ } A \vee B &:= (t_2 =_0 0 \to t_{11} \mathsf{\ mr\ } A) \wedge (t_2 \neq_0 0 \to t_{12} \mathsf{\ mr\ } B)\\
t \mathsf{\ mr\ } A \to B &:= \forall x (x \mathsf{\ mr\ } A \to tx \mathsf{\ mr\ } B)\\
t \mathsf{\ mr\ } \exists y A(y) &:= t_1 \mathsf{\ mr\ } A(t_2)\\
t \mathsf{\ mr\ } \forall y A(y) &:= \forall y (ty \mathsf{\ mr\ } A(y))
\end{aligned}$$

Remark 1. The type of the term t such that $t \mathsf{\ mr\ } A$ is determined solely by the logical form of A.

Fact 4 (Soundness of modified realizability)**.** *From a derivation of $A_1, A_2, \ldots, A_n \vdash B$ in HA$^\omega$, we can construct a term t in the variables $x_1, x_2, \ldots, x_n$ such that $x_1 \mathsf{\ mr\ } A_1, x_2 \mathsf{\ mr\ } A_2, \ldots, x_n \mathsf{\ mr\ } A_n \vdash t \mathsf{\ mr\ } B$ in HA$^\omega$.*

Fact 5. *For any A an instance of AC$^\omega$, there is a term t such that $\vdash t \mathsf{\ mr\ } A$ in HA$^\omega$.*

Theorem 1. *HA$^\omega$+AC$^{\rho\sigma}$+DNS$^\rho$ proves the modified realizability interpretation of DNS$^\rho$.*

Proof: A term t mr-interprets DNS$^\rho$ if and only if, by definition of mr, the following three chains of equivalences hold inside HA$^\omega$:

$$t \mathsf{\ mr\ } \mathrm{DNS}^\rho \tag{1}$$
$$\forall y([y \mathsf{\ mr\ } \forall n^\rho \neg\neg A(n)] \to [ty \mathsf{\ mr\ } \neg\neg\forall n^\rho A(n)]) \tag{2}$$

and

$$y \mathsf{\ mr\ } \forall n^\rho \neg\neg A(n) \tag{3}$$
$$\forall n^\rho (yn \mathsf{\ mr\ } \neg\neg A(n)) \tag{4}$$
$$\forall n^\rho \forall z([z \mathsf{\ mr\ } \neg A(n)] \to [ynz \mathsf{\ mr\ } \bot]) \tag{5}$$
$$\forall n^\rho \forall z \neg[z \mathsf{\ mr\ } \neg A(n)] \tag{6}$$
$$\forall n^\rho \forall z \neg[\forall u(\{u \mathsf{\ mr\ } A(n)\} \to \{zu \mathsf{\ mr\ } \bot\})] \tag{7}$$
$$\forall n^\rho \forall z \neg\forall u \neg\{u \mathsf{\ mr\ } A(n)\} \tag{8}$$
$$\forall n^\rho \neg\forall u \neg\{u \mathsf{\ mr\ } A(n)\} \tag{9}$$
$$\forall n^\rho \neg\neg\exists u\{u \mathsf{\ mr\ } A(n)\} \tag{10}$$

and

$$ty \mathsf{\ mr\ } \neg\neg\forall n^\rho A(n) \tag{11}$$
$$\forall q([q \mathsf{\ mr\ } \neg\forall n^\rho A(n)] \to [tyq \mathsf{\ mr\ } \bot]) \tag{12}$$
$$\forall q \neg[q \mathsf{\ mr\ } \neg\forall n^\rho A(n)] \tag{13}$$
$$\forall q \neg\forall r([r \mathsf{\ mr\ } \forall n^\rho A(n)] \to [qr \mathsf{\ mr\ } \bot]) \tag{14}$$
$$\forall q \neg\forall r \neg\forall n^\rho\{rn \mathsf{\ mr\ } A(n)\} \tag{15}$$
$$\neg\forall r \neg\forall n^\rho\{rn \mathsf{\ mr\ } A(n)\} \tag{16}$$
$$\neg\neg\exists r \forall n^\rho\{rn \mathsf{\ mr\ } A(n)\} \tag{17}$$

Then, the implication $(10) \to (17)$ can be proved using DNS$^\rho$ and AC$^{\rho\sigma}$, where the type σ is determined by the logical complexity of the formula $A(n)$. ■

Corollary 1. *If the theory PA$^\omega$+AC$^\omega$ proves A, then the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ realizes $\neg\neg A$ with a term t extracted by modified realizability from the original proof. Hence, proofs of arithmetical formulas in Classical Analysis (PA$^\omega$+AC^0) can be interpreted inside the theory HA$^\omega$+AC^0+DNS^0 by modified realizability.*

Proof: This follows by applying double-negation translation (Fact 1), then eliminating AC$^\omega{}_\bot$ from the context in favor of AC$^\omega$+DNS$^\omega$ (Fact 3), then applying Soundness for mr (Fact 4), eliminating the mr-translation of AC$^\omega$ by Fact 5, and eliminating mr-translated DNS$^\omega$ in favor of non-translated DNS$^\omega$ and AC$^\omega$ (Theorem 1). In the end, one uses Fact 2 to replace $A^\bot$ by the intuitionistically stronger $\neg\neg A$. ■

In the rest of this section, we will show that the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ is constructive, i.e. that HA$^\omega$+AC$^\omega$+DNS$^\omega$ satisfies certain closure rules characteristic of HA$^\omega$ alone.

We will need the following "with truth" variant of modified realizability from [16], [8].

Definition 3. The interpretation called *modified realizability with truth* (mrt) is obtained when the clause for implication from Definition 2 is replaced by

$$t \mathsf{\ mrt\ } A \to B := \forall x (x \mathsf{\ mrt\ } A \to tx \mathsf{\ mrt\ } B) \wedge (A \to B).$$

We need mrt- because of the following fact that does not hold for simple mr-realizability.

Fact 6 ([8])**.** *For any formula A, we have that, in HA$^\omega$, $\vdash (t \mathsf{\ mrt\ } A) \to A$.*

Corollary 2. *From a derivation in HA$^\omega$+AC$^\omega$+DNS$^\omega$ of $A_1, A_2, \ldots, A_n \vdash B$, one can construct a term t in the variables $x_1, x_2, \ldots, x_n$ such that, in HA$^\omega$+AC$^\omega$+DNS$^\omega$, $x_1 \mathsf{\ mrt\ } A_1, x_2 \mathsf{\ mrt\ } A_2, \ldots, x_n \mathsf{\ mrt\ } A_n \vdash t \mathsf{\ mrt\ } B$.*

Proof: Theorem 1 allows us (by Theorem 3.4.5 of [12] and Theorem 3.5 of [16]) to obtain a soundness theorem for mr- and mrt-interpretation, for the extension HA$^\omega$+AC$^\omega$+DNS$^\omega$ of HA$^\omega$. ■

Corollary 3. *The theory HA$^\omega$+AC$^\omega$+DNS$^\omega$ (and, in particular, also HA$^\omega$+AC^0+DNS^0) satisfies:*

- *the* Existence Property (EP)*: if A is a formula (even an open one) such that $\Gamma \vdash \exists x^\tau A(x)$ then there exists a term t^τ, with free variables $\vec{x}$, such that $\vec{x} \mathsf{\ mrt\ } \Gamma \vdash A(t)$;*
- *the* Existence Property (EP^0) for closed arithmetical formulas $\exists x^0 A(x)$*: if $\vdash \exists x^0 A(x)$, then there exists $n \in \mathbb{N}$ such that $\vdash A(\overline{n})$, where $\overline{n}$ denotes the representation of n inside HA$^\omega$;*
- *the* Disjunction Property (DP^0) for closed arithmetical formulas $A \vee B$*: if $\vdash A \vee B$, then either $\vdash A$ or $\vdash B$;*
- *the* Weak Church Rule (WCR^0) for closed arithmetical formulas $\forall x^0 \exists y^0 A(x, y)$*: if $\vdash \forall x^0 \exists y^0 A(x, y)$, then there exists a total recursive function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n \in \mathbb{N}$, we have that $\vdash A(\overline{n}, \overline{f(n)})$.*

Proof: The proof follows the ones of Corollary 5.24 of [8] and 1.11.7 of [12].

EP is obtained by Corollary 2 and Fact 6.

EP^0 follows by EP and the fact that for every closed term t of type 0 we can find $n \in \mathbb{N}$ such that $\vdash t =_0 \overline{n}$.

DP^0 is proved like EP^0, using the mrt-interpretation of $A \vee B$,

$$\vdash (t =_0 0 \to s \mathsf{\ mrt\ } A) \wedge (t \neq_0 0 \to s \mathsf{\ mrt\ } B),$$

and the fact that, for any $n \in \mathbb{N}$, either $\overline{n} =_0 0$ or $\overline{n} \neq_0 0$.

For WCR, note first that the theory HA$^\omega$+AC$^\omega$+DNS$^\omega$, like HA$^\omega$, is recursively axiomatizable, that is, there exists a recursive predicate $\mathrm{Proof}(k,l)$ formalizing the fact that $k \in \mathbb{N}$ is a code for a derivation of the formula coded by $l \in \mathbb{N}$.

Let $g(n) = \min_m \mathrm{Proof}(j_1 m, \ulcorner A(\overline{n}, \overline{j_2 m})\urcorner)$, where j_1 and j_2 are the projections of some surjective pairing function. By its definition, g is a partial recursive function.

Now, given $\vdash \forall x^0 \exists y^0 A(x,y)$ and $n \in \mathbb{N}$, we obtain $\vdash \exists y^0 A(\overline{n}, y)$, and by EP^0 we obtain $m \in \mathbb{N}$ such that $\vdash A(\overline{n}, \overline{m})$. We proved that, for every n, there exists m s.t. $\vdash A(\overline{n}, \overline{m})$. This shows that the function g is *total* recursive. We may now take $f(n) := j_2(g(n))$ and by definition we have that, for any n, $\vdash A(\overline{n}, \overline{f(n)})$. ■

Remark 2. By the work of Joan Rand Moschovakis [17], we know that the theory HA$^\omega$+AC^0+DNS^0+MP+GC$_1$, where GC$_1$ is a generalization of Brouwer's Continuity Principle, satisfies the following version of Church's Rule, where the recursivity of the function f is proven *inside* the theory,

$$\vdash \forall x^0 \exists y^0 A(x,y) \Rightarrow \vdash \exists z^0 \forall x^0 \exists u^0 (T(z,x,u) \wedge A(x, U(u))). \quad (\mathrm{CR}^0)$$

That result, obtained by a variant of Kleene's number-realizability could probably also be obtained for the restriction HA$^\omega$+AC^0+DNS^0 without MP and GC$_1$.

A. *HA$^\omega$+AC^0+DNS^0 as a distinct variety of Constructive Mathematics*

In contrast to the constructive closure properties satisfied, the theory HA$^\omega$+AC^0+DNS^0 has some surprising meta-mathematical properties that we review in this subsection. Let us first consider the following fact due to Kreisel [18].

Fact 7. *The following forms of DNS are intuitionistically equivalent:*

$$\forall x \neg\neg A(x) \to \neg\neg \forall x A(x) \quad (18)$$
$$\neg \forall x A(x) \to \neg \forall x \neg\neg A(x) \quad (19)$$
$$\neg\neg \forall x (A(x) \vee \neg A(x)) \quad (20)$$
$$\neg[\forall x \neg\neg A(x) \wedge \neg \forall x A(x)] \quad (21)$$

The variant (20) of DNS^0 allows to immediately deduce the *double-negation* of Bishop's *Limited Principle of Omniscience (LPO)* [19]: given $f : \mathbb{N} \to \{0,1\}$,

$$\exists n f(n) = 1 \vee \neg \exists n f(n) = 1, \quad (\mathrm{LPO})$$

or, equivalently,

$$\exists n f(n) = 1 \vee \forall n f(n) = 0.$$

Related to this fact [20], for $A(x) := \exists y T(x,x,y)^3$, from DNS^0 one obtains that it is not the case that the Halting Problem can not be solved,

$$\neg\neg \forall x (\exists y T(x,x,y) \vee \neg \exists y T(x,x,y)). \quad (\neg\neg\mathrm{HP})$$

[3] $T(x,z,y)$ is Kleene's primitive recursive predicate formalizing the fact that the program coded by x, returns with result coded by y, when run on input coded by z. Kleene's function $U(y)$ is used to extract the actual output from the coded result y.

Given the fact that the (single) negation of HP is intuitionistically provable from Church's Thesis [21], this means that DNS^0 refutes Church's Thesis, like the one formalized by

$$\forall x^0 \exists y^0 A(x,y) \to \exists z^0 \forall x^0 \exists u^0 (T(z,x,u) \wedge A(x, U(u))). \quad (\mathrm{CT}^0)$$

Fact 8. *DNS^0 implies $\neg$CT^0.*

This had already been noticed by Gödel [4], reported by Kreisel [22], and also appears in later print (3.5.20 of [12]).

From Corollary 3 and the above proofs of $\neg\neg$LPO, $\neg\neg$HP, and $\neg$CT$_0$, it follows that the system HA$^\omega$+AC^0+DNS^0 is a variety of Constructive Mathematics [19] similar to Brouwer's Intuitionism (that proves $\neg$CT^0), but distinct from Russian Constructivism (that postulates CT$_0$ and proves $\neg$LPO) and from Classical Mathematics (that proves $\neg$CT$_0$, but does not have the Existence Property). Note, however, that, when considered at higher types, DNS$^\omega$, contradicts with certain continuity principles [23].

IV. Dialectica interpretation in presence of DNS

The functional "Dialectica" interpretation was proposed by Gödel [4], [5], [6], before modified realizability was developed, as a way to make precise the somewhat vague notion of construction in the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic implication. The Dialectica interpretation is a realizability technique for intuitionistic Arithmetic (implemented as a program extraction feature in the Minlog proof assistant), that also plays a fundamental role in establishing the consistency of *classical* Mathematics. Namely, by Spector's extension of HA$^\omega$ by an equation schema called Bar Recursion (BR), and by the double-negation translation, Classical Analysis (PA$^\omega$+AC^0) can be interpreted in HA$^\omega$+BR. As mentioned in the Introduction, the key in this interpretation is to Dialectica-interpret DNS^0 by BR.

In this section, we notice that the standard method for interpreting Classical Analysis can be simplified, by omitting the double-negation translation phase. That is, we show that the Dialectica interpretation in presence of DNS^0 already has the double-negation translation built-in.

Definition 4. The term t *interprets the formula A by Dialectica*, if, for all terms s, the relation $|A|_s^t$ holds. This relation is defined by recursion on the complexity of A by the following clauses:

$$\begin{aligned}
|P|_s^t &:= P \qquad \text{if } P\text{-prime}, t = \mathrm{tt}, s = \mathrm{tt}\\
|A \wedge B|_s^t &:= |A|_{s_1}^{t_1} \wedge |B|_{s_2}^{t_2}\\
|A \vee B|_s^t &:= (t_2 =_0 0 \to |A|_{s_1}^{t_{11}}) \wedge (t_2 \neq_0 0 \to |B|_{s_2}^{t_{12}})\\
|A \to B|_s^t &:= |A|_{t_2 s_1 s_2}^{s_1} \to |B|_{s_2}^{t_1 s_1}\\
|\exists x A(x)|_s^t &:= |A(t_2)|_s^{t_1}\\
|\forall x A(x)|_s^t &:= |A(s_2)|_{s_1}^{t s_2}
\end{aligned}$$

Fact 9 (Soundness of Dialectica interpretation [12])**.** *From a derivation of $A_1, A_2, \ldots, A_n \vdash B$ in HA$^\omega$, we can construct a term t in the variables $x_1, x_2, \ldots, x_n$ such that $\forall y |A_1|_y^{x_1}, \forall y |A_2|_y^{x_2}, \ldots, \forall y |A_n|_y^{x_n} \vdash \forall y |B|_y^t$ in HA$^\omega$.*

Fact 10 ([12])**.** *For any A an instance of AC$^\omega$, there is a term t such that $\vdash \forall y |A|_y^t$ in HA$^\omega$.*

Fact 11 ([8]). *From the stability (decidability) of quantifier-free formulas in HA$^\omega$, and the fact that the $|\cdot|$ translation eliminates quantifiers, we have that, in HA$^\omega$,*

$$\vdash |\neg\neg A|_s^t \leftrightarrow |A|_{s_{12}(t_2 s_1 s_2)_1 (t_2 s_1 s_2)_2}^{(t_2 s_1 s_2)_1}.$$

Spector's result can now be stated in the following form.

Fact 12 (Theorem 11.6 of [8]). *If $\vdash A$ in PA$^\omega$+AC^0, then there exists a term t such that $\vdash \forall y |A^\perp|_y^t$ in HA$^\omega$+BR.*

The following theorem is an improvement of the previous fact, that nonetheless uses the same old technology.

Theorem 2. *If $\vdash A$ in PA$^\omega$+AC^0, then there exists a term t such that $\vdash \forall y |A|_{y_{12}(t_2 y_1 y_2)_1 (t_2 y_1 y_2)_2}^{(t_2 y_1 y_2)_1}$ in HA$^\omega$+BR.*

Proof: Starting from a derivation

$$\text{PA}^\omega\text{+AC}^0 \vdash A,$$

by double-negation translation (Fact 1), we obtain

$$\text{HA}^\omega\text{+AC}^0{}_\perp \vdash A^\perp.$$

From Fact ... and Fact 2, we get

$$\text{HA}^\omega\text{+AC}^0\text{+DNS}^0 \vdash \neg\neg A.$$

Now, by Soundness of Dialectica (Fact 9), facts 10 and 11, and the fact that DNS^0 is provable in HA$^\omega$+BR, we obtain

$$\text{HA}^\omega\text{+BR} \vdash \forall y |A|_{y_{12}(t_2 y_1 y_2)_1 (t_2 y_1 y_2)_2}^{(t_2 y_1 y_2)_1}$$

as needed. ∎

Remark 3. The *quantifier-free* axiom of choice,

$$\forall x^\rho \exists y^\sigma A_0(x,y) \to \exists f^{\rho\to\sigma} \forall x^\rho A_0(x, f(x)), \qquad \text{(QF-AC)}$$

where A_0 is quantifier free, can be included among the premises in both Fact 12 [12] and Theorem 2, thanks to the fact that its interpretation depends only on the Dialectica-interpretability of Markov's Principle at higher-types (which is available). We have not done so, in order not to confuse the reader with two choice axioms that partially overlap.

Remark 4. While the interpretation of Classical Analysis from Section III takes place in the theory HA$^\omega$+AC^0+DNS^0, Spector's interpretation takes place in HA$^\omega$+BR without AC^0.

V. Applications of derivability of DNS by delimited control operators

The original motivation for studying the meta-mathematical properties of DNS is our previous work [24], [11] on the logical system MQC$_+$ that derives DNS using the shift and reset delimited control operators of Danvy and Filinski [9], [10], an abstraction developed in the theory of programming languages.

In this section, we propose to formalize some arguments from Section III that use DNS, by delimited control operators directly. For that purpose, consider the first-order logic given in natural deduction form in Table I. The usual natural deduction rules for intuitionistic logic are given in the upper box. They have a turnstile annotated by a diamond $_\diamond$, which is a wild-card that stands either for nothing (empty annotation), or stands for $_\perp$. In any case, for the intuitionistic rules, the wild-card stands for the same thing both in the premises and in the conclusion of the rule.

The non-intuitionistic rules are given in the lower box. There is the reset rule, that can be used with an empty $_\diamond$ or with $_\diamond$ being $_\perp$. In the former case, the role of reset is to begin a classical proof of $\perp$; in the later case, there is no logical meaning to the reset rule, but there is computational meaning to it, coming from the operational semantics for shift and reset. There is also the shift rule, that allows to use a principle similar to double-negation elimination from classical logic, but only inside a derivation that is ultimately proving $\perp$, as guaranteed by a reset rule which must have set the annotation to $\perp$ before.

For more explanations and the computational behavior of MQC$_+$, please refer to [11].

We now give proof terms that formalize the proofs of the four alternative versions of DNS from Fact 7:

$$\lambda h.\lambda k.\#k(\tilde{\lambda} x.\mathcal{S}k'.hxk') \qquad (22)$$

$$\lambda k.\lambda h.\#k(\tilde{\lambda} x.\mathcal{S}k'.hxk') \qquad (23)$$

$$\lambda k.\#k(\tilde{\lambda} x.\mathcal{S}k'.k'(\mathrm{inr}\,(\lambda a.k'(\mathrm{inl}\,a)))) \qquad (24)$$

$$\lambda h.\#(\mathrm{snd}\,h)(\tilde{\lambda} x.\mathcal{S}k.(\mathrm{fst}\,h)xk) \qquad (25)$$

We can see that the proof term (24), that also formalizes the proof or $\neg\neg$HP, awaits to receive a proof term for $\neg$HP, and, once it gets it, it uses $\#$ to prove $\perp$ by classical logic: to the obtained proof term it supplies a classical proof of HP,

$$\tilde{\lambda} x.\mathcal{S}k'.k'(\mathrm{inr}\,(\lambda a.k'(\mathrm{inl}\,a))) \\ : \forall x(\exists y T(x,x,y) \vee \neg\exists y T(x,x,y)).$$

To give the proof term for $\neg$CT$_0$, we only need a formalized proof of the implication CT$_0 \to \neg$HP. This proof is intuitionistic and, in principle, it could be formalized in HA$^\omega$, following, for example, [21].

This technique can be used to *refute* in MQC$_+$+HA$^\omega$ *any* property P that implies $\neg$HP or $\neg$LPO. Assume that such a proof term

$$r : P \to \neg\text{HP}$$

is given. Then,

$$\lambda p.\#r(\tilde{\lambda} x.\mathcal{S}k'.k'(\mathrm{inr}\,(\lambda a.k'(\mathrm{inl}\,a)))) : \neg P.$$

Since reducing a problem to the non-solvability of the Halting Problem is a common proof method, this means that all such results can be seen in a new light in the constructive logic HA$^\omega$+AC^0+DNS^0.

VI. Future and related work

In the future, we would like to extend the modified realizability interpretation to the whole of the system MQC$_+$+HA$^\omega$. This would allow us to extract a realizer for $\neg\neg$AC$^\omega$ by delimited control operators. Currently, it is not known whether DNS$^\omega$ captures exactly the additional provability power that MQC$_+$ has over intuitionistic predicate logic.

Usually, modified realizability and the Dialectica interpretation are developed using Hilbert-style systems. In this paper, we used natural deduction. For detailed development of mr-

$$\frac{(a:A)\in\Gamma}{\Gamma\vdash_{\diamond} a:A}\,\mathrm{Ax} \qquad \frac{\Gamma\vdash_{\diamond} p:A_1 \qquad \Gamma\vdash_{\diamond} q:A_2}{\Gamma\vdash_{\diamond} (p,q):A_1\wedge A_2}\,\wedge_I \qquad \frac{\Gamma\vdash_{\diamond} p:A_1\wedge A_2}{\Gamma\vdash_{\diamond} \mathrm{fst}(p):A_i}\,\wedge_E^1 \qquad \frac{\Gamma\vdash_{\diamond} p:A_1\wedge A_2}{\Gamma\vdash_{\diamond} \mathrm{snd}(p):A_i}\,\wedge_E^2$$

$$\frac{\Gamma\vdash_{\diamond} p:A_i}{\Gamma\vdash_{\diamond} \mathrm{inl}(p):A_1\vee A_2}\,\vee_I^1 \qquad \frac{\Gamma\vdash_{\diamond} p:A_1\vee A_2 \qquad \Gamma,a_1:A_1\vdash_{\diamond} q_1:C \qquad \Gamma,a_2:A_2\vdash_{\diamond} q_2:C}{\Gamma\vdash_{\diamond} \textsf{case}\ p\ \textsf{of}\ (a_1.q_1\|a_2.q_2):C}\,\vee_E$$

$$\frac{\Gamma\vdash_{\diamond} p:A_i}{\Gamma\vdash_{\diamond} \mathrm{inr}(p):A_1\vee A_2}\,\vee_I^2 \qquad \frac{\Gamma,a:A_1\vdash_{\diamond} p:A_2}{\Gamma\vdash_{\diamond} \lambda a.p:A_1\to A_2}\,\to_I \qquad \frac{\Gamma\vdash_{\diamond} p:A_1\to A_2 \qquad \Gamma\vdash_{\diamond} q:A_1}{\Gamma\vdash_{\diamond} pq:A_2}\,\to_E$$

$$\frac{\Gamma\vdash_{\diamond} p:A(x) \qquad x\text{-fresh}}{\Gamma\vdash_{\diamond} \tilde{\lambda}x.p:\forall xA(x)}\,\forall_I \qquad \frac{\Gamma\vdash_{\diamond} p:\forall xA(x)}{\Gamma\vdash_{\diamond} pt:A(t)}\,\forall_E \qquad \frac{\Gamma\vdash_{\diamond} p:A(t)}{\Gamma\vdash_{\diamond} (t,p):\exists x.A(x)}\,\exists_I$$

$$\frac{\Gamma\vdash_{\diamond} p:\exists x.A(x) \qquad \Gamma,a:A(x)\vdash_{\diamond} q:C \qquad x\text{-fresh}}{\Gamma\vdash_{\diamond} \textsf{dest}\ p\ \textsf{as}\ (x.a)\ \textsf{in}\ q:C}\,\exists_E \qquad \frac{\Gamma\vdash_{\diamond} p:\bot}{\Gamma\vdash_{\diamond} \mathrm{efq}(p):A}\,\bot_E$$

$$\frac{\Gamma\vdash_{\bot} p:\bot}{\Gamma\vdash_{\diamond} \#p:\bot}\,\#\ (\text{“reset”}) \qquad \frac{\Gamma,k:A\to\bot\vdash_{\bot} p:\bot}{\Gamma\vdash_{\bot} \mathcal{S}k.p:A}\,\mathcal{S}\ (\text{“shift”})$$

TABLE I. NATURAL DEDUCTION SYSTEM WITH PROOF TERMS FOR MQC_+

and D-interpretation in the context of natural deduction, the reader may refer to [25].

In the context of Constructive Reverse Mathematics, Wim Veldman has shown that, in presence of Markov's Principle, the Double-negation Shift is equivalent to Open Induction on Cantor space [26]. Based on his work, in unpublished work [27], Keiko Nakata and the author show how to prove the principle of Open Induction on Cantor Space using delimited control operators.

In [28], Paulo Oliva revisits Spector's bar recursive interpretation of DNS. In particular, he remarks that the finitary version of DNS, in which "$\forall$" is replaced by a finite conjuction, is provable intuitionistically.

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